Technical Report No. 32-440

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for Multireceiver Communications
Through the Rician Multichannel

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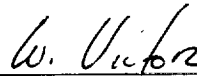CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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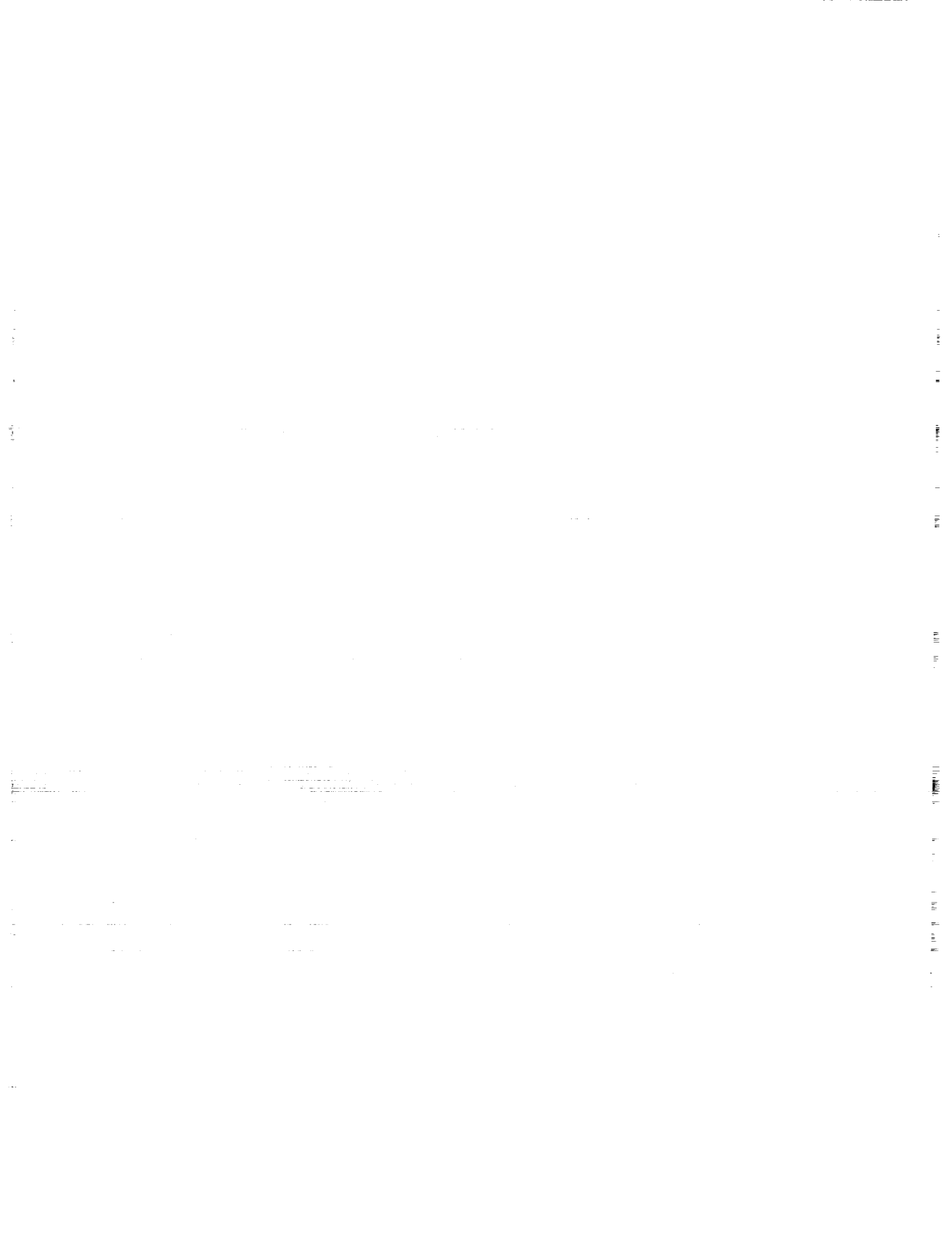PASADENA, CALIFORNIA

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PREFACE

This paper presents results of one phase of research carried out at the Jet Propulsion Laboratory, Pasadena, California (JPL), under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

ABSTRACT

Presented in this paper are ideal asymptotic performance charac-
teristics for two types of multireceivers–the coherent adaptive multi-
receiver and the noncoherent multireceiver. The transmitter for the
system selects for transmission one of two equiprobable, equal-energy
correlated waveforms. The selected waveform is transmitted into the
Rician fading multichannel with additive Gaussian noise superposed at
the receiver end. The multichannel may take on one of four forms, i.e.,
it may change from one of a completely random nature to that of a
completely fixed nature. All intermediate forms are permissible.

It is shown that multichannel communication links containing fixed
or specular components have, inherent in their performance character-
istics, an affinity for exponential factors. This attraction is independent
of the receiver termination, regardless of whether or not it be coherent
or noncoherent. On the other hand, multichannel communication links
operating through multichannels of a completely random nature have
an affinity for inverse factors, i.e., system performance increases or the
probability of error decreases inversely with some power of the signal-
to-noise ratio. Finally, it is shown that multichannel communication
links containing combinations of purely fixed and random modes pos-
sess, in their performance characteristics, combinations of these inverse
and exponential factors. These conclusions are important, because
they indicate the rapidity with which system performance changes as
the propagation medium characteristics change from a completely ran-
dom nature to that of a fixed structure.

At low error probabilities, the coherent multireceiver, which per-
forms channel measurements, yields a 6-db improvement in signal-to-
noise ratio over a non-coherent multireceiver which is ignorant of the
channel state. Strikingly enough, this improvement is realized for both
the completely random multichannel and the random multichannel
supporting small fixed components. At low signal-to-noise ratios, an
8-db improvement may be realized. These improvement factors may
be considered as upper bounds for other types of fading represented
by the Rician fading model, i.e., the non-fading case and the multi-
channel model which possesses small random components and large
specular components.

I. INTRODUCTION

The demand for greater speed and reliability in binary data communications requires the physical realization of "optimum" receivers. Interest is increasing in those receivers designed to take into consideration (by channel measurement techniques) the random characteristics of the propagation medium. Typical examples of this type of medium are ionospheric radio links operating below the maximum usable frequency, tropospheric and ionospheric scatter links, lunar relay links, long range space communication channels employing slow data rates, and possibly orbital-chaff channels.

As a result of these requirements, an effort is made to compare the ideal asymptotic performance characteristics for two types of transmitter-receiver terminations that are challenged to perform reliably through four types of practical random multichannels.[1] The receiver terminations are assumed to be the coherent and non-coherent multireceivers. The coherent multireceiver is presumed to be capable of performing ideal measurements on the multichannel's gain and phase characteristics, whereas the non-coherent multireceiver is not required to perform measurements.[2] In either case, it is presumed that the multireceivers are operating synchronously with the modulation at the transmitter.

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[1] The term "multichannel" is borrowed from a recent paper by Price (Ref. 1). It is used here so as to include both diversity reception and resolvable multipath reception.

[2] Channel measurement techniques are not discussed here but Kailath (Ref. 2) discusses the problems of making detailed channel measurements.

The four types of multichannels considered are:

(1) The Rician Fading Multichannel

(2) The Rayleigh Fading Multichannel

(3) The Fixed Mode Multichannel

(4) The Mixed-Mode Multichannel

The practical importance of considering these multichannels is fully established in a host of propagation literature (Refs. 3 and 4).

Another salient feature here is that, in the coherent termination, the asymptotic results are directly applicable to adaptive communication links (those which are capable of performing measurements on certain channel characteristics while simultaneously transmitting information and continuously adjusting modes of operation so as to optimize performance with respect to some criterion chosen *a priori*). Examples of operative adaptive communication links are the Rake (Ref. 5) and Kineplex (Ref. 6) systems. These analyses point out the theoretical importance of making measurements on the communication media and also indicate whether or not the improvement in performance is sufficient to warrant the cost and complexity of adding the additionally required measuring equipment.

A recent paper by Price (Ref. 1) has pointed out the improvement in system performance obtained by performing non-ideal measurements on the additive white Gaussian noise channel terminated in two types of adaptive receivers.

II. DETAILED SYSTEM MODEL

Figure 1 illustrates the generic model of the multichannel communication link under consideration.

At the transmitter, a choice is made between two narrow-band signals, $s_1(t)$ and $s_2(t)$, and one is selected for transmission into the multichannel. It is assumed that both signals are zero outside some time interval, e.g., $0 \leq t \leq T$. Within this time interval, however, the waveshapes may be simple pulsed sine-waves of different frequencies or they may be of considerable complexity. In the diversity situation, the selected signal is transmitted over M statistically independent channels, whereas in the resolvable multipath case the receiver hears M distinct echoes as a result of the dispersive transmission medium. The two signals may be either uncorrelated or highly correlated in the coherent termination. The only other restrictions placed upon the signals aside from the requirement of finite time durations, are that they have equal probabilities of transmittal and that they have the same energy.

The signal chosen for transmission first passes through the random multichannel. After passing through the random multichannel the transmitted signal is further perturbed by additive noise, which is assumed to be white, stationary, Gaussian, and statistically independent of the random multichannel. The resultant signal $y(t)$ is called the observed data for that particular transmission.

The chore of the multireceiver, which is presumed to have stored replicas of the two possible transmitted waveforms (but does not know which was actually sent), is to determine, with minimum error probability, which waveform was selected for transmission. It does this by operating on $y(t)$, and it can be shown (Ref. 7) that this operation consists of computing the a posteriori probabilities, $p(s_1|y)$ and $p(s_2|y)$, of the two signals, and choosing the one with the larger probability. That is, the receiver always conjectures that the transmitted signal is the one which seems most probable on the basis of analysis of the received waveform and the amount of information provided it by the multichannel identifier. It is sufficient to say that the multichannel identifier computes from the observed data modification signals necessary for optimizing the receiver structure.
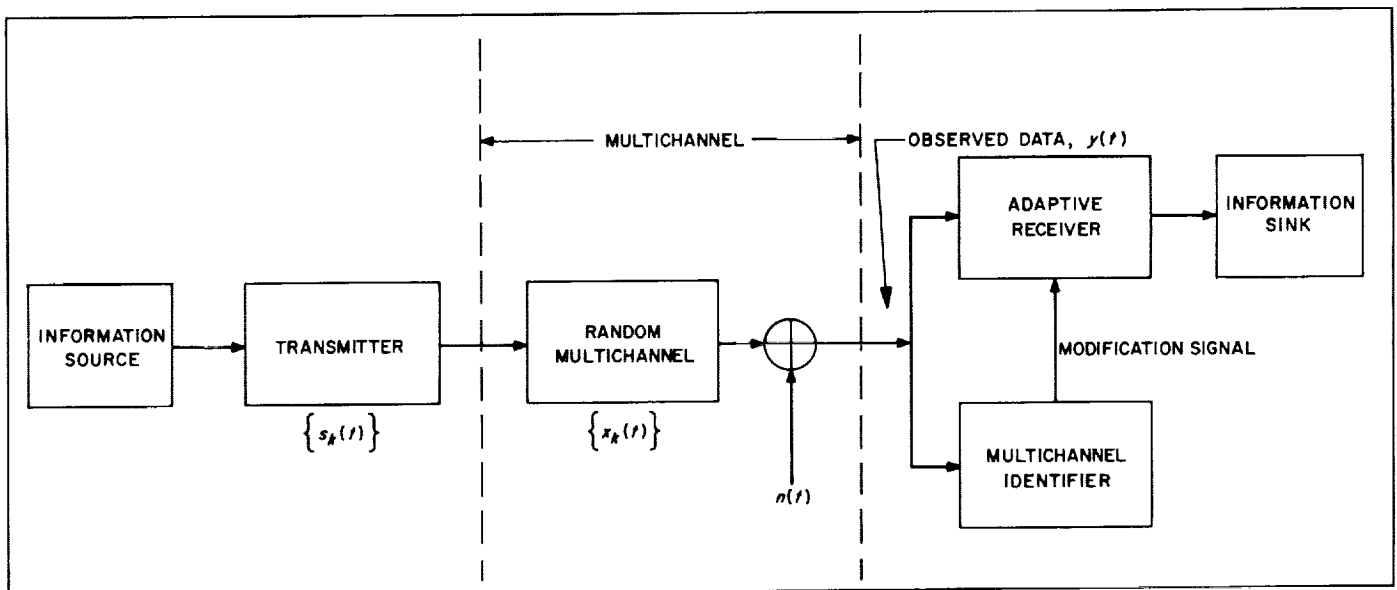

Fig. 1. Communication link

III. THE SIGNALS

It has been assumed that there are two narrowband signaling waveforms at the transmitter, which are zero outside the interval $0 \leq t \leq T$. These signals may be conveniently represented by using Woodward's (Ref. 8) notation, i.e., representing the two signals as products of complex low-pass modulation waveforms and cisoidal carriers

$$s_1(t) = z_1(t) \, e^{j\omega_0 t}$$

$$s_2(t) = z_2(t) \, e^{j\omega_0 t} \tag{1}$$

where ω_0 is a suitably defined carrier frequency. Another basic communication parameter is defined as:

$$\lambda = \left| \frac{1}{2E} \int_0^T s_1(t) \, s_2 *(t) \, dt \right| \tag{2}$$

The quantity λ may be described as the relative cross-correlation between signals, and E is the common signal energy. This will sufficiently describe the signals.

IV. THE MULTICHANNEL

Figure 2 represents an expanded view of the random multichannel. The multichannel model is perhaps best understood by describing what happens to a signal which passes through it. If it is presumed that $s_k(t)$ was transmitted with signal parameters $a = 1$, $\theta = 0$, and $\tau = 0$, then the output of the ith channel may be written (Ref. 9) as

$$x_k(t) = a_i s_k(t - \tau_i) \, e^{j(\omega_0 t - \theta_i)}$$

$$= s_k(t - \tau_i) \left[\underbrace{\alpha_i e^{-j\delta_i}}_{\substack{\text{specular} \\ \text{component}}} + \underbrace{s_i e^{j\varepsilon_i}}_{\substack{\text{random} \\ \text{component}}} \right] e^{j\omega_0 t} \tag{3}$$

That is, each propagation mode is characterized by three random quantities: a_i, the gain of the ith channel, θ_i, the phase characteristic of the ith channel, and τ_i, the modulation delay of the ith channel. It is necessarily assumed that the channel states do not appreciably change over the transmission interval. Hence, the first order joint density distribution $p(a_i, \theta_i)$ sufficiently describes the fading medium (Ref. 9). In any given case it is assumed that the modulation delay characteristic $\bar{\tau} = (\tau_1, \tau_2, \cdots \tau_M)$ is known to the multireceiver whereas the multichannel gain $\bar{a} = (a_1, a_2, \cdots a_M)$ and the multichannel phase characteristic $\bar{\theta} = (\theta_1, \theta_2, \cdots \theta_M)$, if known to the multireceiver, are obtained as a result of measurement. Briefly, it is sufficient to say that each propagation mode is envisioned as a composite of a fixed component of strength α_i and phase-shift δ_i, and is a scatter-like random component

with completely random phase and Rayleigh-distributed amplitude with a mean square value of $2\sigma^2$. The joint distribution for a_i and θ_i is given (Ref. 9) by

$$p(a_i, \theta_i) = \frac{a_i}{2\pi\sigma^2} \exp \left[- \frac{a_i^2 + \alpha_i^2 - 2\alpha_i a_i \cos(\theta_i - \delta_i)}{2\sigma^2} \right]$$

$$0 \leq a_i \leq \infty$$

$$0 \leq (\theta_i - \delta_i) \leq 2\pi \tag{4}$$

$$= 0 \text{ elsewhere}$$

Experimental justification indicating the validity of Eq. (4) is given fully in the propagation literature, for example, see Refs. 3 and 4. The expression typifies channel conditions for both ionospheric and tropospheric radio links operating above and below the maximum usable frequency. It may be used, as a good approximation, for representing the received signal strength and carrier phase shift of a signal received from a tumbling satellite or orbital-chaff. An additional use for this type of channel would be in communications via the lunar surface. Radar returns from the lunar surface indicate that most of the power reflected from the Moon is returned by specular reflection even though some of the signal power is returned from the lunar surface out to the limb (Refs. 10 and 11). The latter accounts for the fading of the reflected waves and corresponds to a large $\alpha_i^2 / 2\sigma^2$ in Eq. (4). Hence, such a model depicts communications via the lunar surface.

The three channel parameters α_i, σ, and δ_i of Eq. (4) may be given physical interpretations. The quantity α_i may be considered to be the strength of the fixed (specular) component in the ith channel, and $2\sigma^2$, the mean squared value of the random component in the ith channel. For convenience purposes, $\gamma_i^2 = \alpha_i^2 / 2\sigma^2$ is defined as the ratio of the average energy received via the fixed channel component to the average energy received via the random component. Moreover, it is assumed that the noise sources are statistically independent, and all channels have noises of the same root-mean-square value. Thus, the multichannel model is sufficiently general to represent several types of diversity systems such as frequency, time and space, or the resolvable multipath situation. For a more thorough description of the multichannel the reader is referred to Turin (Ref. 9).

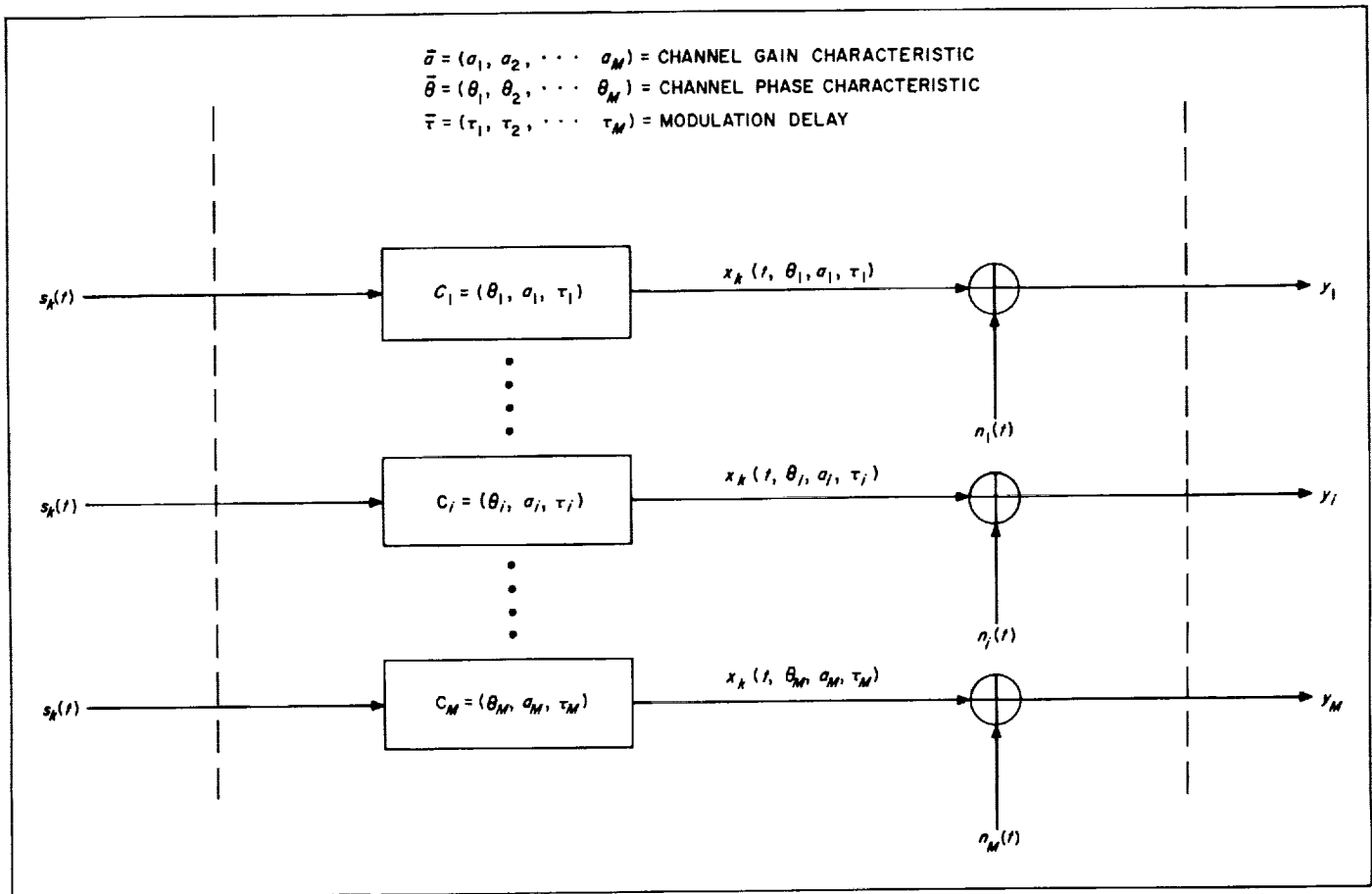$\bar{a} = (a_1, a_2, \cdots a_M) = $ CHANNEL GAIN CHARACTERISTIC
$\bar{\theta} = (\theta_1, \theta_2, \cdots \theta_M) = $ CHANNEL PHASE CHARACTERISTIC
$\bar{\tau} = (\tau_1, \tau_2, \cdots \tau_M) = $ MODULATION DELAY

Fig. 2. A multilink channel

V. THE MULTIRECEIVER

Figure 3 depicts the ith branch of the multireceivers which are connected to the system just described. In the coherent case, the phase measurement is inserted into the observed data. Electronically speaking, this may be done in other ways. Following the phase insertion equipment are filters matched to the signals stored at the transmitter. (The phantom lines denote the other branches of the ith receiver). The matched filter outputs are then weighted in accordance with the gain of the ith channel and sampled at the end of the signal interval. After phase shifting, filtering, weighting and sampling, the outputs of all filters matched to signal 1 are summed and compared with the summed values obtained from the filters matched to signal 2. Under the foregoing assumptions, these operations can be shown to be optimum in the *a posteriori* probability computing sense (Refs. 9 and

12), and interestingly enough, are equivalent to the combining technique outlined by Brennan (Ref. 13). In the coherently terminated multireceiver, the final decision is made in favor of the signal which gives rise to the largest sum.

For the non-coherent multireceiver (Fig. 2), the first portion of the ith receiver consists of a pair of filters again matched to signals 1 and 2. The outputs of these filters are followed by square-law envelope detectors which are sampled at the conclusion of the signaling interval. The final decision is ultimately made as before, i.e., the samples are summed and the one yielding the larger sum is chosen as the signaling state to be used at the transmitter. If the channel is completely random, this combining method is optimum under the foregoing assumptions
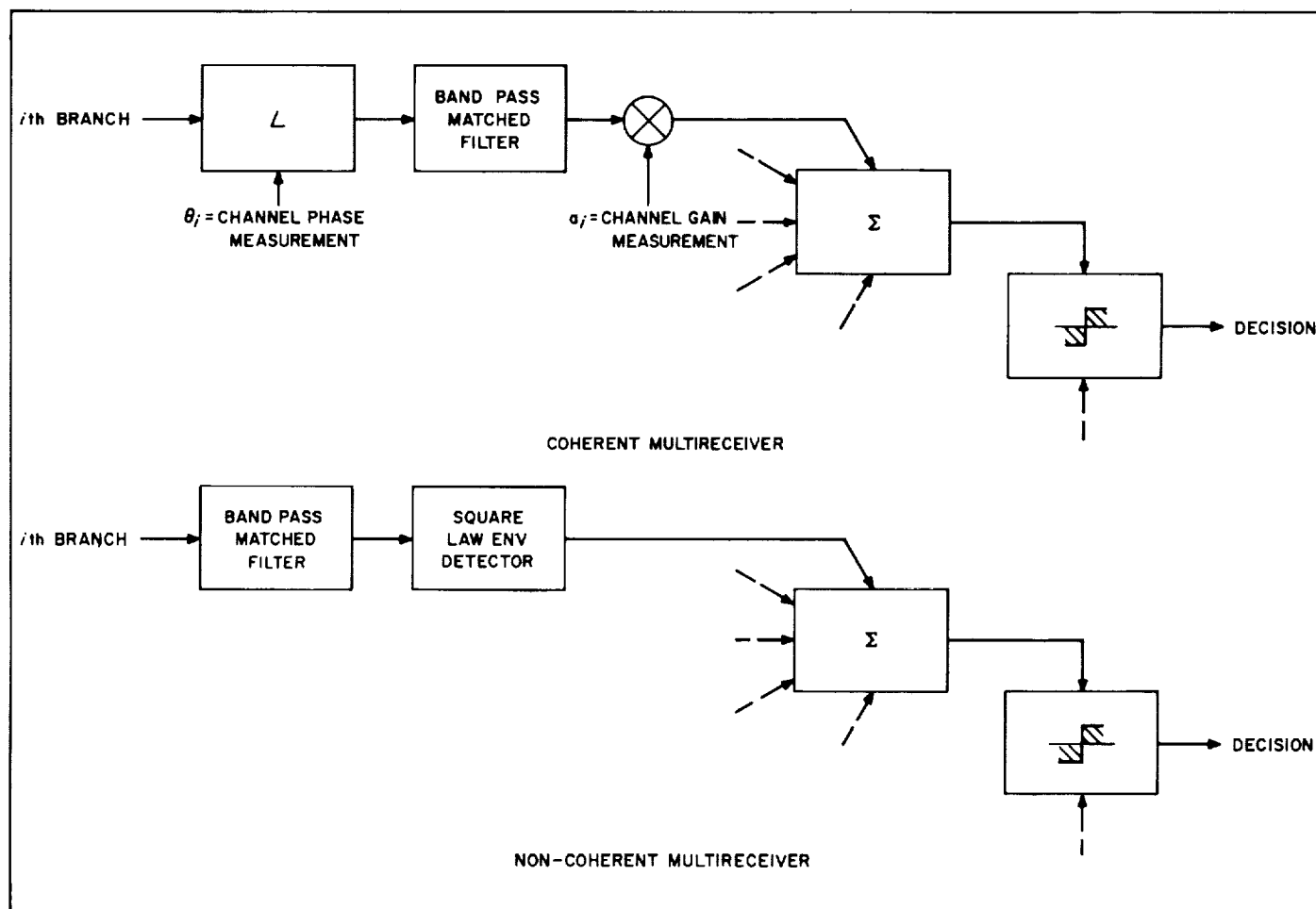

Fig. 3. Coherent and non-coherent multireceiver

(Refs. 9 and 14). Such is not the case if the multichannel contains fixed components. The only difference is that the non-linear characteristic of the receiver must change. For a discussion of this characteristic, see Ref. 9. Having described the optimality of the multireceivers under the presumption of either perfect measurement or no measurement, we now turn to the problem of receiver performance characteristics.

VI. ASYMPTOTIC CHARACTERISTICS OF THE ADAPTIVE COHERENT MULTIRECEIVER FOR VARIOUS MULTICHANNEL CONDITIONS

Certain asymptotic performance characteristics have been analyzed and are presented in Appendices A and B.

Listed below are the important communication link parameters that have arisen in the analysis:

λ signal cross-correlation coefficient

ρ_i signal-to-noise ratio of the fixed component

M Multichannel order

β signal-to-noise ratio of the random component

$\gamma_i^2 = \dfrac{\alpha_i^2}{2\sigma^2} = \dfrac{\text{strength of fixed component}}{\text{mean square value of random component}}$

$\bar{a} = (a_1, a_2 \cdots a_M)$ multichannel gain characteristic

$\bar{\theta} = (\theta_1, \theta_2, \cdots \theta_M)$ multichannel phase characteristic

$\bar{\tau} = (\tau_1, \tau_2, \cdots \tau_M)$ modulation delay characteristic

$P_{E_n}^{\bar{\theta}, \bar{a}, \bar{\tau}}(M)$ System error rate given measurements on $\bar{\theta}, \bar{a},$ and $\bar{\tau}$. (Coherent multireceiver)

$P_{E_n}^{\bar{\tau}}(M)$ System error rate given measurements on τ. (Non-coherent multireceiver)

$n = 1$ refers to the Rician fading multichannel

$n = 2$ refers to the Rayleigh fading multichannel

$n = 3$ refers to the fixed mode multichannel

$n = 4$ refers to the mixed mode multichannel

The purpose of this section is to present the result for the completely random multichannel and then proceed through the other results presuming that the multichannel characteristics change. This displays the rapidity with which the performance of the coherent multireceiver changes as the propagation characteristics change.

For low error probabilities, i.e., $\beta \gg 1$, the completely random multichannel supporting equally reliable propagation modes is

$$P_{E_1}^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{1}{2} \binom{2M}{M} \left[\frac{1}{2\beta(1-\lambda)} \right]^M \qquad (5)$$

The quantity $\binom{2M}{M}$ is the binomial coefficient $\dfrac{(2M)!}{(M!)^2}$.

From this expression it is obvious that phase-reversal binary-signaling techniques yield a three db improvement in signal-to-noise ratio over ordinary orthogonal FSK signaling techniques. This is not surprising since it has been assumed that coherent reception is possible. If λ is set equal to zero, the result agrees with that obtained by Pierce (Ref. 14).

If the Rician fading multichannel is now considered with small specular components, i.e., $(\beta > \rho)$, it is possible to show that[3] (see Appendix A)

$$P_{E_1}^{\theta,a,\tau}(M) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \prod_{i=1}^{M}\exp\left[-\gamma_i^2\right]$$

(6)

which agrees with Eq. (5) if $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$.

Clearly the effect of multichannel specular components is to insert exponential factors in the system performance expression, whereas the random nature of the multichannel introduces inverse factors (factors which decrease with an increasing signal-to-noise ratio of the random component). Obviously, these exponential factors may remarkably increase system performance. In particular, the ratio of Eq. (6) to Eq. (5) is

$$\frac{P_{E_1}^{\theta,a,\tau}(M)}{P_{E_2}^{\theta,a,\tau}(M)} = \prod_{i=1}^{M}\exp-\left[\gamma_i^2\right] \leq 1$$

(7)

If the fixed channel components are equally reliable, then

$$P_{E_1}^{\theta,a,\tau}(M) = \exp\left[-M\gamma^2\right]P_{E_2}^{\theta,a,\tau}(M)$$

(8)

for $\beta > 1$.

It is now assumed that the multichannel state changes to the conditions where the fixed components are larger

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[3]In all equations containing the γ_i^2 factors, it is possible to replace the product of M distinct terms by one of $M - N$ terms where $M > N$, i.e., a multichannel may be considered which possesses N Rayleigh fading propagation modes of mean strength $2\ \sigma^2$, and $M - N$ Rician modes. This is what is meant by the mixed mode multichannel, and consequently, $n = 4$.

than the random components, i.e., $\rho > \beta$. In this case, asymptotically:

$$P_{E_1}^{\theta,a,\tau}(M) \sim$$

$$\frac{\prod\limits_{i=1}^{M}\exp\left[-\gamma_i^2\left(\frac{\beta(1-\lambda)}{2+\beta(1-\lambda)}\right)\right]}{\sqrt{2\pi\sum\limits_{i=1}^{M}\rho_i(1-\lambda)}}\left[\frac{2}{2+\beta(1-\lambda)}\right]^{M-1}$$

(9)

where $\gamma_i^2\ \beta = \rho_i$. From Eq. (9), note that the exponential factor is now the predominant feature in the asymptotic characteristic. The inverse factor due to the random channel component is still present, however.

As a final condition, it is assumed that the multichannel modes become fixed. In this case $\beta = 0$, and $\rho > 0$.

$$P_{E_2}^{\theta,a,\tau}(M) \sim \frac{\prod\limits_{i=1}^{M}\exp\left[-\frac{\rho_i(1-\lambda)}{2}\right]}{\sqrt{2\pi\sum\limits_{i=1}^{M}\rho_i(1-\lambda)}}$$

(10)

The salient feature of this result is that system performance increases approximately exponentially with increasing signal-to-noise ratio. Recall from Eq. (5) that for the completely random multichannel, system performance increased inversely with the Mth power of β. There is a large variance between this and the exponential behavior shown for the fixed mode multichannel. If $M = 1$ in Eq. (10), the result agrees with that of Turin (Ref. 15). Equation (10) is recognizable as the asymptotic expansion

for the error function evaluated at $\sqrt{\sum\limits_{i=1}^{M}\rho_i(1-\lambda)}$. This

provides a convenient check with the same result derived by another method in Appendix B.

VII. ASYMPTOTIC CHARACTERISTICS OF THE NON-COHERENT MULTIRECEIVER FOR VARIOUS MULTICHANNEL CONDITIONS

This discussion is begun with the assumptions that the multichannel is completely random and that transmitted signals are orthogonal. System performance utilizing correlated signals has been derived. However, the asymptotic results have not been computed. These results will be reported later.

For the completely random multichannel, asymptotically:

$$P_{E_2}^{\bar{T}}(M) \sim \frac{1}{2} \binom{2M}{M} \left[\frac{1}{\beta} \right]^M \tag{11}$$

for $\beta > 1$. The quantity $\binom{2M}{M}$ is the binomial coefficient

$\frac{(2M!)}{(M!)^2}$. Pierce (Ref. 14) was the first to derive this result.

Needless to say, system performance increases or error probability decreases inversely with the signal-to-noise ratio of the random component. Note the absence of the exponential factors.

Suppose now, that due to changing propagation characteristics, the multichannel possesses small specular components of distinct strength α_i. For this situation, with $\beta > \rho$, (see Appendix C)

$$P_{E_1}^{\bar{T}}(M) \sim \frac{1}{2} \binom{2M}{M} \left[\frac{1}{\beta} \right]^M \prod_{i=1}^{M} \exp\left[-\gamma_i^2 \right] \tag{12}$$

This is the asymptotic result for the Rician multichannel. If $\gamma_i^2 = 0$ for all $i = 1, 2, \cdots M$, Eq. (11) agrees with Pierce's result. From Eq. (12) it may be concluded that the exponential factors are existent in the system error rate as a

result of the specular channel components, and that the inverse factor is due to the random channel component. Obviously, these exponential factors could remarkably increase system performance. Taking the ratio of Eq. (12) to Eq. (11) yields

$$\frac{P_{E_1}^{\bar{T}}(M)}{P_{E_2}^{\bar{T}}(M)} = \prod_{i=1}^{M} \exp\left[-\gamma_i^2 \right] \leq 1 \tag{13}$$

which is the same ratio obtained under similar conditions for the coherent case.

Assuming that further changes in the propagation characteristics introduce large specular components, the following bound for $\beta < \rho$, has been established:

$$P_{E_1}^{\bar{T}}(M) < \left[\frac{1}{2+\beta} \right]^M \prod_{i=1}^{M} \exp\left[-\frac{\gamma_i^2 \beta}{2+\beta} \left(1 - \frac{1}{2+\beta} \right) \right] \tag{14}$$

where $\gamma_i^2 \beta = \rho_i$. For this multichannel condition, the exponential factors dominate system performance while the inverse factor appears to be least significant.

As a final multichannel state, it is assumed that the random components are zero, i.e., $\beta = 0$, or the propagation modes are fixed. The asymptotic result for this situation is

$$P_{E_3}(M) < \frac{1}{2^M} \prod_{i=1}^{M} \exp - \left[\frac{\rho_i}{4} \right] \tag{15}$$

Again, the performance characteristic possesses only exponential factors (as would be expected). For $M = 1$, it can be seen that Eq. (15) is three db from the exact curve.

VIII. PERFORMANCE COMPARISON FOR THE TWO MULTIRECEIVER TERMINATIONS

At this point, attention is directed toward the rather important problem of system comparison. This problem is of interest because the results indicate the gain in performance obtainable by using the additionally required receiver equipment or, for equivalent operating characteristics, how much saving in transmitter power may be realized by using the more sophisticated equipment. Both areas under question are of great concern in space communications, where transmitter power capabilities are constrained as a result of weight requirements.

The comparison is begun by a consideration of the completely random multichannel. Equations (5) and (11) are rewritten, i.e., for $\beta \gg 1$

$$P_{E_1}^{\bar{\sigma},\bar{a},\bar{\tau}}(M) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \tag{16}$$

$$P_{E_2}^{\bar{\tau}}(M) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^M \tag{17}$$

A comparison of these two expressions shows that the coherent adaptive multireceiver is capable of yielding, for any M, a 6-db improvement in signal-to-noise ratio over the noncoherent multireceiver. Moreover, it has been shown (Ref. 16) for low signal-to-noise ratios that the coherent adaptive multireceiver outperforms the noncoherent multireceiver by an 8-db factor. This corresponds to a considerable saving in transmitted power.

Assuming that the propagation medium introduces small specular components, Eqs. (6) and (12) are compared:

$$P_{E_1}^{\bar{\sigma},\bar{a},\bar{\tau}}(M) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{2\beta(1-\lambda)}\right]^M \prod_{i=1}^M \exp\left[-\gamma_i^2\right] \tag{18}$$

$$P_{E_2}^{\bar{\tau}}(M) \sim \frac{1}{2}\binom{2M}{M}\left[\frac{1}{\beta}\right]^M \prod_{i=1}^M \exp\left[-\gamma_i^2\right] \tag{19}$$

The same conclusions may be reached for this mixed mode multichannel as was for the completely random multichannel. It should be noted, however, that for equiv-alent transmitter powers, system performance would be superior as a result of the multichannel specular components.

For the multichannel with large specular components, Eqs. (9) and (19) are compared:

$$P_{E_1}^{\bar{\sigma},\bar{a},\bar{\tau}}(M) \sim \frac{\prod_{i=1}^M \exp\left[-\gamma_i^2\left(\frac{\beta(1-\lambda)}{2+\beta(1-\lambda)}\right)\right]}{\sqrt{2\pi\sum_{i=1}^M \rho_i(1-\lambda)}}\left[\frac{2}{2+\beta(1-\lambda)}\right]^{M-1} \tag{20}$$

$$P_{E_1}^{\bar{\tau}}(M) < \left[\frac{1}{2+\beta}\right]^M \prod_{i=1}^M \exp\left[-\frac{\gamma_i^2\,\beta}{2+\beta}\left(1-\frac{1}{2+\beta}\right)\right] \tag{21}$$

where $\rho_i = \gamma_i^2\,\beta$. By comparing these two equations, it is difficult to state the exact amount by which the adaptive multireceiver would outperform the noncoherent multireceiver. The results do, however, indicate important performance trends, and it is intuitive that the coherent multireceiver would outperform the noncoherent multireceiver. The precise amount is currently being determined from the exact performance expressions (Ref. 17).

Finally, for the fixed mode multichannel, Eqs. (10) and (15) would be compared, and it would again be concluded that the adaptive multireceiver's performance is superior to that of the noncoherent multireceiver. It is important to note that for the fixed mode multichannel or the Rician multichannel, the amount of improvement cannot exceed that realized when the multichannel is completely random. Hence, the improvement obtainable is upper bounded by 8 db in the low signal-to-noise ratio region, and upper bounded by 6 db in the region of high signal-to-noise ratios.

IX. CONCLUSIONS

This Report has presented ideal asymptotic performance characteristics for a transmitter challenged to operate through a random multichannel which may assume one of four states. The results are applicable to two types of multireceiver terminations: the adaptive coherent multireceiver and the noncoherent multireceiver.

It has been assumed in the foregoing analysis that the energy associated with the signals stored at all M transmitters has been equal. If, on the other hand, an attempt had been made to derive the asymptotic formulas under the assumption of distinct signal energies, difficulty would have been encountered in averaging the multichannel gain. This suggests that an *a priori* signal-energy distribution may exist, which will minimize the error probability. Intuitively, one would expect that most of the energy should be transmitted into those channels which are not encountering severe or deep fades. This is in contrast to the fixed multichannel case where the asymptotic error probability depends only upon the total transmitted energy. Price (Ref. 1) has noted this result. The manner in which the energies should be distributed among the various channels is yet to be determined.

To make the analysis tractable, the mean squared values of all scatter components have been assumed equal. If such an assumption is not invoked, it appears that only the analysis for the completely random multichannel may be carried out. In reality, it is hardly conceivable that all scatter components would be of equivalent mean strength.

Finally, it can be pointed out that for the coherent termination, the transmitted signal should be negatively correlated in order to yield a minimum error probability with respect to a variation of possible signaling waveforms. On the other hand, for the noncoherent termination, specification of the value of λ which minimizes the error probability $(M > 1)$ has not yet been solved.

ACKNOWLEDGEMENT

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APPENDIX A

Probability of Error — Coherent Multichannel System $(\beta > \rho)$

For the coherent multichannel system, it may be shown that the conditional probability of error (Refs. 16 and 18) is given by:

$$P_E(X) = P_E(a_1, a_2, \cdots a_M) = \frac{1}{\sqrt{\pi}} \int_A^\infty e^{-v^2} \, dy$$

(A-1)

where

$$A = \sqrt{\sum_{i=1}^M \frac{a_i^2 R(1-\lambda)}{2}} = \sqrt{\frac{XR(1-\lambda)}{2}}$$

$$R = \frac{E}{N_0} = \frac{\text{Signal energy}}{\text{Double-sided noise-power density}}$$

(A-2)

λ = signal cross-correlation coefficient (see Eq. 2)

$$X = \sum_{i=1}^M a_i^2$$

To find $p(X)$, a result obtained in Helstrom (Ref. 7) is used. After an appropriate change in notation:

$$p(X) = \frac{1}{2\sigma^2} \left(\frac{X}{P}\right)^{\frac{M-1}{2}} \exp\left[-\frac{X+P}{2\sigma^2}\right] \cdot I_{M-1}\left[\frac{1}{\sigma^2}\sqrt{PX}\right]$$

$$; X \geq 0 \qquad (A-3)$$

$$= 0 \text{ elsewhere}$$

where $P = \sum_{i=1}^M a_i^2$, and $I_{M-1}(x)$ is the modified Bessel function of the first kind of order $M-1$. For small values of x,

$$I_{M-1}(x) \sim \left(\frac{x}{2}\right)^{M-1}\left(\frac{1}{\Gamma(M)}\right)$$

(A-4)

Thus, for small specular components, Eq. (A-3) may be rewritten:

$$p(X) \sim \frac{X^{M-1}}{(2\sigma^2)^M \, \Gamma(M)} \exp\left[-L\right]$$

(A-5)

where $L = P/2\sigma^2 = \sum_{i=1}^M \gamma_i^2$.

The average error rate may be obtained from Eq. (A-1) by averaging over the variable X, i.e.,

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) = \int_0^\infty p(X) \, P_E(X) \, dX$$

(A-6)

Using Eqs. (A-1) and (A-5) in Eq. (A-6):

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{1}{\sqrt{\pi}} \int_0^\infty p(X) \, dX \int_A^\infty e^{-v^2} \, dy$$

(A-7)

If the orders of integration are now changed (which is justifiable), Eq. (A-7) becomes

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-v^2} \, dy \int_0^{v^2/W} p(X) \, dX$$

(A-8)

where $W = \sqrt{\frac{R}{2}(1-\lambda)}$. Substituting Eq. (A-5) into Eq. (A-8) yields

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{e^{-L}}{\sqrt{\pi} \, M! \, (2\sigma^2)^M} \int_0^\infty \frac{y^{2M}}{W^M} e^{-v^2} \, dy$$

(A-9)

This easily integrates with respect to y, giving

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{e^{-L}}{2(M!)} \frac{[1 \cdot 3 \cdot 5 \cdots (2M-1)]}{[2 \, \sigma^2 R(1-\lambda)]^M}$$

(A-10)

and may be rewritten as

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{e^{-L}}{M!}$$

$$\frac{[1 \cdot 3 \cdot 5 \cdots (2M-1)][2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdots 2(M-1)]}{[2\sigma^2 R(1-\lambda)]^M \, 2^M \, (M-1)!}$$

(A-11)

which is recognizable as

$$P_E^{\bar{\theta}, \bar{a}, \bar{\tau}}(M) \sim \frac{e^{-L} \, (2M-1)!}{M! \, (M-1)!} \left[\frac{1}{2\beta(1-\lambda)}\right]^M$$

(A-12)

Substituting the value for L in Eq. (A-12) gives Eq. (6). If L is set equal to zero, Eq. (5) is produced.

APPENDIX B

Probability of Error — Coherent Multichannel System $(\beta < \rho)$

Consider now the multichannel which possesses small random components and large specular components. First, $p(X)$ is rewritten in terms of the new variable $X = \sigma^2 t^2$. Thus, $p(X)$ becomes

$$p(t) = t \left(\frac{t}{a}\right)^{M-1} \exp\left[-\frac{t^2 + a^2}{2}\right] I_{M-1}(at) \ ; \ t > 0$$

(B-1)

where $a = \frac{\sqrt{P}}{\sigma}$. Thus, Eq. (A-1) becomes

$$P_E(t) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{dt^2}}^{\infty} e^{-z^2/2} \, dz$$

(B-2)

where $d = \sigma^2 R(1 - \lambda)$. Using the asymptotic expansion for the error function,

$$P_E(t) \sim \frac{1}{\sqrt{2\pi dt^2}} \exp\left[-\frac{dt^2}{2}\right]$$

(B-3)

for $dt^2 > 0$. The error rate expression becomes

$$P_E^{\bar{\theta},\bar{a},\bar{\tau}}(M) \sim \int_0^{\infty} p(t) P_E(t) \, dt$$

(B-4)

which gives, upon substituting Eqs. (B-1) and (B-3) into Eq. (B-4),

$$P_E^{\bar{\theta},\bar{a},\bar{\tau}}(M) \sim \frac{1}{\sqrt{2\pi d}} \int_0^{\infty} \left(\frac{t}{a}\right)^{M-1} \exp\left[-\frac{(1 + d)t^2 + a^2}{2}\right]$$
$$\times I_{M-1}(at) \, dt$$

(B-5)

Integration of Eq. (B-5) may be performed by using the well known result (Ref. 19):

$$\int_0^{\infty} e^{-a^2x^2} x^{\mu-1} I_v(bx) dx = \frac{b^v \Gamma\left(\frac{\mu + v}{2}\right)}{2^{v+1} a^{\mu+v} \Gamma(v + 1)}$$
$$\times F\left[\frac{\mu + v}{2}, v + 1, \frac{b^2}{4a^2}\right]$$

(B-6)

where $F(a, b, c)$ is the hypergeometric function defined by

$$F(a, b, c) = 1 + \frac{a}{b} \frac{c}{1!} + \frac{a(a+1)}{b(b+1)} \frac{c^2}{2!} + \cdots$$

(B-7)

Substituting this into Eq. (B-5) gives

$$P_E^{\bar{\theta},\bar{a},\bar{\tau}}(M) \sim \frac{1}{\sqrt{2\pi d}} \frac{1}{2^M} \left[\frac{2}{1 + d}\right]^{M-\frac{1}{2}} \exp[-L]$$
$$\times \frac{\Gamma\left(\frac{2M - 1}{2}\right)}{\Gamma(M)} F\left[\frac{2M - 1}{2}, M, \frac{L}{1 + d}\right]$$

(B-8)

The asymptotic expansion for the hypergeometric function $F(a, c, z)$ is now used:

$$F(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \frac{e^z}{z^{c-a}} \left[1 + 0\left(\frac{1}{z}\right)\right]$$

(B-9)

And,

$$P_E^{\bar{\theta},\bar{a},\bar{\tau}}(M) \sim \frac{e^{-L\left(\frac{d}{1+d}\right)}}{\sqrt{4\pi Ld}} \left[\frac{1}{1 + d}\right]^{M-1}$$

The definitions for d, L, β, and ρ provide Eq. (9).

APPENDIX C

Probability of Error — Noncoherent Multichannel System $(\beta > \rho)$

The exact performance expression for the noncoherent multireceiver is given by

$$P_{\overline{E}}^{\overline{i}}(M) = \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \binom{M+m-1}{m}$$

$$\times \left(\frac{1+\beta}{2+\beta}\right)^m F\left(-m, M, -\frac{L\beta}{(1+\beta)(2+\beta)}\right) \quad \text{(C-1)}$$

where $L = \sum_{i=1}^{M} \gamma_i^2$ and $F(a, b, z)$ is the confluent hypergeometric function defined by Eq. (B-7).

Because of limited space, the derivation of Eq. (C-1) is not provided. However, the details may be found in a Jet Propulsion Laboratory Technical Report, Ref. (17). For large random components or small specular components, the hypergeometric function may be approximated by unity when $\beta \gg 1$. Under such channel conditions,

$$P_{\overline{E}}^{\overline{i}}(M) \sim \left[\frac{1}{2+\beta}\right]^M \exp\left[-L\right] \sum_{m=0}^{M-1} \binom{M+m-1}{m}$$

$$\times \left(\frac{1+\beta}{2+\beta}\right)^m \quad \text{(C-2)}$$

Using a technique developed by Pierce (Ref. 14),

$$\frac{1+\beta}{2+\beta} = 1 - \frac{1}{2+\beta}$$

is written, and a binomial expansion is made of each term in the sum. After the terms are rearranged, this procedure yields:

$$P_{\overline{E}}^{\overline{i}}(M) \sim \left[\frac{1}{2+\beta}\right]^M \exp\left[-L\right] \sum_{m=0}^{M-1}$$

$$\frac{(-1)^m (2M-1)!}{(M-1)!\,(M-m-1)!\,m!\,(m+M)} \left(\frac{1}{2+\beta}\right)^m \quad \text{(C-3)}$$

At large signal-to-noise ratios (small error probability), the first term in the sum predominates, and

$$P_{\overline{E}}^{\overline{i}}(M) \sim \frac{1}{2} \exp\left[-L\right] \binom{2M}{M}\left[\frac{1}{\beta}\right]^M \quad \text{(C-4)}$$

Substituting for L provides Eq. (12).

APPENDIX D

Probability of Error—Noncoherent Multichannel System $(\beta < \rho)$

In the derivation of Eq. (14), it is assumed that the multichannel possesses large specular components such that $\rho_i > \beta$. Using the first term of the asymptotic expansion for $F(a, c, z)$ (see Eq. (B-9)) and Eq. (C-1),

$$P_E^{\bar{\tau}}(M) \sim \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\right] \sum_{m=0}^{M-1} \frac{1}{m!} \times \left[\left(\frac{1+\beta}{2+\beta}\right)x\right]^m \quad \text{(D-1)}$$

where $x = \dfrac{L\beta}{(1+\beta)(2+\beta)}$

Then,

$$1 \leq \sum_{m=0}^{M-1} \frac{1}{m!}\left[\left(\frac{1+\beta}{2+\beta}\right)x\right]^m < e^{\frac{1+\beta}{2+\beta}x} \quad \text{(D-2)}$$

and the following upper bound is determined:

$$P_E^{\bar{\tau}}(M) < \left[\frac{1}{2+\beta}\right]^M \exp\left[-\frac{L\beta}{2+\beta}\left(1 - \frac{1}{2+\beta}\right)\right] \quad \text{(D-3)}$$

Substituting for L provides Eq. (14). Setting $\beta = 0$,

$$P_E^T(M) < \left[\frac{1}{2}\right]^M \exp\left[-\sum_{i=1}^M \frac{\rho_i}{4}\right] \quad \text{(D-4)}$$

which is Eq. (15).